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Nonoscillatory Solutions of a Class of nth-Order Linear Differential Equations

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We consider the differential equation of the form

$$y^{(n)} + py = 0, \tag{E}$$

where p is continuous and of one sign on an interval $[a, \infty)$. Equation (E) has been extensively studied by a number of authors [1–13], and it is well known that the behaviors of solutions of (E) depend strongly on the parity of n and the sign of p. Consequently, we consider the following four cases of (E):

(i) n even, $p \geqslant 0$,

(ii) n odd, $p \geqslant 0$,

(iii) n even, $p \leqslant 0$,

(iv) n odd, $p \leqslant 0$.

Equation (E) satisfying, for example, condition (i) is denoted (E_i). Likewise, (E_{ii}), (E_{iii}), and (E_{iv}) denote Eq. (E) satisfying (ii), (iii), and (iv), respectively.

A nontrivial solution of (E) is said to be *nonoscillatory* on $[a, \infty)$ if it does not have infinitely many zeros on $[a, \infty)$. Equation (E) is said to be *nonoscillatory* on $[a, \infty)$ if every nontrivial solution of (E) is nonoscillatory on $[a, \infty)$. It is evident from this definition that we may assume $y > 0$ on $[b, \infty)$, for some $b \geqslant a$, if y is a nonoscillatory solution of (E). Let S_0 be the set of bounded nonoscillatory solutions of (E), let S_k, $k = 1, 2,..., n - 1$, be the set of nonoscillatory solutions y with the property that

$$\lim_{x\to\infty}(y(x)/x^{k-1}) > K,$$

for some positive constant K, and

$$\lim_{x\to\infty}(y(x)/x^{k}) = 0,$$

and let S_n be the set of nonoscillatory solutions such that

$$\lim_{x\to\infty}(y(x)/x^{n-1}) > K,$$

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for some positive constant K. Define

$$\begin{aligned}
\mathscr{S} &= \bigcup_{i=0}^{j} S_{2i} && \text{for } (\mathrm{E_i}) \text{ if } n = 2j, \\
&= \bigcup_{i=0}^{j} S_{2i+1} && \text{for } (\mathrm{E_{ii}}) \text{ if } n = 2j + 1, \\
&= \bigcup_{i=0}^{j-1} S_{2i+1} && \text{for } (\mathrm{E_{iii}}) \text{ if } n = 2j, \\
&= \bigcup_{i=0}^{j} S_{2i} && \text{for } (\mathrm{E_{iv}}) \text{ if } n = 2j + 1.
\end{aligned}$$

Under the condition that

$$\int^{\infty} x^{n-1} \,|\, p(x)| \; dx = \infty, \tag{1}$$

it was recently proved that S_0 is empty (i.e., every nonoscillatory solution is unbounded) for ($\mathrm{E_i}$) and ($\mathrm{E_{iv}}$), and that S_1 is empty for ($\mathrm{E_{ii}}$) and ($\mathrm{E_{iii}}$) [4, 6]. In this paper we extend these results and establish the following theorem.

THEOREM 1. *If $\int^{\infty} x^{n-1} \,|\, p(x)| \; dx = \infty$, the set $\mathscr{S}$ of nonoscillatory solutions of* (E) *is empty.*

We need a few preliminary results which are repeatedly used in the proof of Theorem 1.

LEMMA 1. *Suppose $y \in C^n[b, \infty)$, $y \geqslant 0$ on $[b, \infty)$,*

$$\lim_{t\to\infty}(y(t)/t^l) = 0,$$

for some integer l, $1 \leqslant l \leqslant n - 1$, and $y^{(n)} \not\equiv 0$ on $[b_1, \infty)$ for every $b_1 \geqslant b$. If $y^{(n)} \leqslant 0$ on $[b, \infty)$, then

$$(-1)^{k+1} y^{(n-k)}(t) > 0, \qquad t \in [b, \infty),$$

for $k = 1, 2,..., n - l$, and also for $k = n - l + 1$ if $n - l$ is even. On the other hand, if $y^{(n)} \geqslant 0$ on $[b, \infty)$, then

$$(-1)^{k} y^{(n-k)}(t) > 0, \qquad t \in [b, \infty),$$

$k = 1, 2,..., n - l$, where the inequality also holds for $k = n - l + 1$ if $n - l$ is odd.

Proof. Consider the case $y^{(n)} \leqslant 0$ on $[b, \infty)$. If $y^{(n-1)}(\alpha) \leqslant 0$ for some $\alpha \in [b, \infty)$, then there exist a positive constant K and a point $\beta \in [\alpha, \infty)$ such that $y^{(n-1)}(t) < -K < 0$, $t \in [\beta, \infty)$. But this implies that $y(t) \to -\infty$ as $t \to \infty$, contradicting the inequality $y \geqslant 0$ on $[b, \infty)$. Thus, $y^{(n-1)}(t) > 0$, $t \in [b, \infty)$. If $l = n - 1$, the proof is complete. Otherwise, $l \leqslant n - 2$, and we need to prove $y^{(n-2)}(t) < 0$, $t \in [b, \infty)$. If $y^{(n-2)}(\alpha) \geqslant 0$ for some $\alpha \in [b, \infty)$, then $y^{(n-2)}(t) > K$, for some positive constant K, on some interval $[\beta, \infty)$. However, this implies that $y(t) > K_1 t^{n-2}$, $t \in [\beta_1, \infty)$, for some constants $K_1 > 0$ and $\beta_1 \geqslant \beta$, which contradicts the asymptotic behavior

$$\lim_{t\to\infty}(y(t)/t^l) = 0,$$

because $l \leqslant n - 2$. Consequently, we have $y^{(n-2)}(t) < 0$, $t \in [b, \infty)$. If $l = n - 2$, the proof is complete; otherwise, we repeat the above arguments to prove $(-1)^{k+1} y^{(n-k)}(t) > 0$, $t \in [b, \infty)$, $k = 1, 2,\ldots, n - l$. If $n - l$ is even, we have for $k = n - l$, $y^{(l)}(t) < 0$, $t \in [b, \infty)$. In this case, we can repeat a preceding argument one more time and conclude that $y^{(l-1)}(t) > 0$, $t \in [b, \infty)$. Proof for the other case is similar.

LEMMA 2. *If Eq.* (E) *has a solution $y \in S_l$ for some l, $1 \leqslant l \leqslant n - 1$, and if $p \not\equiv 0$ on $[a_1, \infty)$ for every $a_1 \geqslant a$, then*

$$(-1)^{k+1} \lambda \left(y^{(n-k)}(t) + (-1)^k \frac{A_l}{(k-1)!} \int_t^x (s-t)^{k-1} s^{l-1} p(s)\, ds\right) > 0, \quad (2)$$

$c \leqslant t \leqslant x < \infty$, $k = 1, 2,\ldots, n - l$, for some constants c and $A_l > 0$, where $\lambda = +1$ if $p \geqslant 0$ and $\lambda = -1$ if $p \leqslant 0$. For ($\mathrm{E_i}$) *and* ($\mathrm{E_{iv}}$) *inequality* (2) *also holds for $k = n - l + 1$ if l is even; and for* ($\mathrm{E_{ii}}$) *and* ($\mathrm{E_{iii}}$), (2) *holds for $k = n - l + 1$ if l is odd. Finally,* (2) *holds for $k = 1$ and $l = n$ if* ($\mathrm{E_i}$) *or* ($\mathrm{E_{ii}}$) *has a solution $y \in S_n$.*

Proof. We prove the lemma for the case ($\mathrm{E_i}$):

$$(-1)^{k+1} \left(y^{(n-k)}(t) + (-1)^k \frac{A_l}{(k-1)!} \int_t^x (s-t)^{k-1} s^{l-1} p(s)\, ds\right) > 0, \quad (3)$$

$c \leqslant t \leqslant x < \infty$, for $k = 1, 2,\ldots, n - l$, for $k = n - l + 1$ if l is even, and for $k = 1$ and $l = n$. Suppose that $p \geqslant 0$ on $[a, \infty)$ and $y \geqslant 0$ on $[b, \infty)$, $b \geqslant a$. Since $y \in S_l$,

$$\lim_{x\to\infty}(y(x)/x^{l-1}) > A_l, \quad (4)$$

for some constant $A_l > 0$, and

$$\lim_{x\to\infty}(y(x)/x^l) = 0. \quad (5)$$

It follows from (4) that $y(x) > A_l x^{l-1}$, $x \in [c, \infty)$, for some $c \geqslant b$, and

$$y^{(n)}(s) = -p(s)\,y(s) < -A_l s^{l-1} p(s) \leqslant 0, \qquad s \in [c, \infty). \tag{6}$$

Integrating (6) from t to x, $c \leqslant t \leqslant x < \infty$, we get

$$y^{(n-1)}(x) < y^{(n-1)}(t) - A_l \int_t^x s^{l-1} p(s)\, ds. \tag{7}$$

On the other hand, we must have $y^{(n-1)}(x) > 0$ by Lemma 1. Thus, we conclude from (7) that

$$y^{(n-1)}(t) - A_l \int_t^x s^{l-1} p(s)\, ds > 0, \qquad c \leqslant t \leqslant x < \infty. \tag{8}$$

If $l = n - 1$, the proof is complete. If $l < n - 1$, we integrate (8) from t to x, $c \leqslant t \leqslant x < \infty$, and obtain

$$\begin{aligned} y^{(n-2)}(x) &> y^{(n-2)}(t) + A_l \int_t^x d\xi \int_\xi^x s^{l-1} p(s)\, ds \\ &= y^{(n-2)}(t) + A_l \int_t^x (s - t)\, s^{l-1} p(s)\, ds. \end{aligned} \tag{9}$$

Since Lemma 1 requires that $y^{(n-2)}(x) < 0$, $x \in [c, \infty)$, we deduce from (9) that

$$y^{(n-2)}(t) + A_l \int_t^x (s - t)\, s^{l-1} p(s)\, ds < 0, \qquad c \leqslant t \leqslant x < \infty.$$

If $l = n - 2$, the proof is complete; otherwise, we repeat a similar argument $n - l$ times and prove (3) for $k = 1, 2, \ldots, n - l$. If l is even, then $n - l$ is even and (3) yields for $k = n - l$,

$$y^{(l)}(t) + \frac{A_l}{(n - l - 1)!} \int_t^x (s - t)^{n-l-1}\, s^{l-1} p(s)\, ds < 0, \qquad c \leqslant t \leqslant x < \infty. \tag{10}$$

In this case, we may use the preceding argument one more time, since the first inequality in Lemma 1 holds for $k = n - l + 1$ and gives $y^{(l-1)}(x) > 0$, $x \in [b, \infty)$. Integrating (10) from t to x, $c \leqslant t \leqslant x < \infty$, we get

$$y^{(l-1)}(x) < y^{(l-1)}(t) - \frac{A_l}{(n - l)!} \int_t^x (s - t)^{n-l}\, s^{l-1} p(s)\, ds, \qquad c \leqslant t \leqslant x < \infty. \tag{11}$$

The inequality $y^{(l-1)}(x) > 0$, $x \in [b, \infty)$, and (11) are incompatible unless

$$y^{(l-1)}(t) - \frac{A_l}{(n - l)!} \int_t^x (s - t)^{n-l}\, s^{l-1} p(s)\, ds > 0,$$

which proves (3) for $k = n - l + 1$ when l is even.

Turning to the proof of (3) for $k = 1$ and $l = n$, we must have $y^{(n-1)} > 0$ on $[b, \infty)$ because otherwise $y(x) \to -\infty$ as $x \to \infty$. Since $y \in S_n$,

$$y^{(n)}(s) = -p(s)\, y(s) < -A_n s^{n-1} p(s), \qquad s \in [c, \infty),$$

for some constants $A_n > 0$ and $c \geqslant b$. When this inequality is similarly integrated, and the result compared with $y^{(n-1)} > 0$ on $[b, \infty)$, we see that

$$y^{(n-1)}(t) - A_n \int_t^x s^{n-1} p(s)\, ds > 0.$$

Proofs for the other cases are similar.

Proof of Theorem 1. We prove that the set $S_0 \cup S_2 \cup \cdots \cup S_n$, $n = 2j$, is empty for (Ei). Proofs for the other cases are similar. S_0 is known to be empty for (Ei) [4, 6]. Assume that (Ei) has a solution $y \in S_{2m}$, for some m, $1 \leqslant m \leqslant n/2$. We apply Lemma 2 to the solution y and obtain for $k = n - 2m + 1$,

$$y^{(2m-1)}(t) - \frac{A_{2m}}{(n-2m)!} \int_t^x (s-t)^{n-2m}\, s^{2m-1} p(s)\, ds > 0, \tag{12}$$

$c \leqslant t \leqslant x < \infty$, for some constants $A_{2m} > 0$ and c. However, the integral

$$I \equiv \int_t^x (s-t)^{n-2m}\, s^{2m-1} p(s)\, ds$$

approaches $+\infty$ as $x \to +\infty$: for any γ, $c \leqslant t \leqslant \gamma \leqslant x$,

$$I \geqslant \left(1 - \frac{t}{\gamma}\right)^{n-2m} \int_\gamma^x s^{n-1} p(s)\, ds \to +\infty,$$

as $x \to +\infty$. Therefore, (12) cannot hold for all x, $c \leqslant t \leqslant x < \infty$; and this contradiction proves that S_{2m}, $1 \leqslant m \leqslant n/2$, is empty for (Ei).

Equation (E) is said to be *disconjugate* on an interval I if no nontrivial solution of (E) has more than $n - 1$ zeros (where the zeros are counted with their multiplicities). It is well known that (E) is disconjugate on $[\alpha, \infty)$, for some α, if the integral $\int^\infty x^{n-1} p(x)\, dx$ is finite [12, 13]. Hence, Theorem 1 has the following corollary.

COROLLARY 1. *If* (E) *has a solution belonging to* $\mathscr{S}$, *it is disconjugate on* $[\alpha, \infty)$, *for some* α.

Note that the explicit form of the coefficient p in (E) is not required to conclude disconjugacy in Corollary 1.

We now proceed to show that the sets $S_0, S_1, \ldots, S_n$ form a partition of the set of nonoscillatory solutions of (E), provided that condition (1) is satisfied. It is evident from the definition of S_k, $k = 0, 1, \ldots, n$, that $S_i \cap S_j = \varnothing$,

$i \neq j$, except for $S_0 \cap S_1$ which consists of bounded solutions y such that $y(x)$ does not approach zero as $x \to \infty$. However, if condition (1) is satisfied, every nonoscillatory solution of (E) either is unbounded or approaches zero [4, 6, 10], i.e., $S_0 \cap S_1$ is empty.

Suppose that y is a nonoscillatory solution such that $y \geqslant 0$ on $[b, \infty)$. We assert that either

$$\lim_{x\to\infty}(y(x)/x^k) = 0,$$

or

$$\lim_{x\to\infty}(y(x)/x^k) > A_k\,,$$

for some positive constant A_k, $k = 0, 1,..., n-1$. To prove this statement, assume that neither is true. Then

$$0 = \liminf_{x\to\infty}(y(x)/x^k) < \limsup_{x\to\infty}(y(x)/x^k) = \xi,$$

and there would exist a constant B, $0 < B < \xi$, such that $y(x)/x^k$ takes on the value B infinitely many times, that is, the function $g_k(x) \equiv y(x) - Bx^k$ has infinitely many zeros on $[b, \infty)$. Consequently, $g_k^{(n-1)} = y^{(n-1)}$ has an infinity of zeros on $[b, \infty)$, $k = 0, 1,..., n-2$. But this is impossible because $y^{(n)} \geqslant 0$ or $y^{(n)} \leqslant 0$ on $[b, \infty)$. Similarly, $g_{n-1}^{(n-1)} = y^{(n-1)} - B(n-1)!$ cannot have an infinity of zeros on $[b, \infty)$. This contradiction proves the assertion.

If the nonoscillatory solution y is bounded on $[b, \infty)$, it belongs to S_0. Assume that y is unbounded. If

$$\lim_{x\to\infty}(y(x)/x^{n-1}) > C,$$

for some positive constant C, then y belongs to S_n. Otherwise, there exists a positive integer $m < n$ such that

$$\lim_{x\to\infty}(y(x)/x^{m-1}) > A_{m-1}\,,$$

for some positive constant A_{m-1}. Let m be the largest such integer. Then the inequality

$$\lim_{x\to\infty}(y(x)/x^m) > A_m$$

cannot hold for any positive constant A_m, and therefore

$$\lim_{x\to\infty}(y(x)/x^m) = 0$$

by the preceding assertion. Hence, $y \in S_m$. This shows that any nonoscillatory solution of (E) belongs to S_k, for some k, $0 \leqslant k \leqslant n$.

Thus, we have proved

THEOREM 2. *If condition* (1) *is satisfied, the sets* $S_0, S_1, \ldots, S_n$ *form a partition of the set of nonoscillatory solutions of* (E).

From Theorems 1 and 2, we readily deduce the following statements.

COROLLARY 2. *If condition* (1) *is satisfied, every nonoscillatory solution of* (E) *belongs to* $\mathscr{S}'$, *where*

$$\begin{aligned} \mathscr{S}' &= \bigcup_{i=0}^{j-1} S_{2i+1} && \textit{for} \quad (\mathrm{E_i}) \textit{ if } n = 2j, \\ &= \bigcup_{i=0}^{j} S_{2i} && \textit{for} \quad (\mathrm{E_{ii}}) \textit{ if } n = 2j + 1, \\ &= \bigcup_{i=0}^{j} S_{2i} && \textit{for} \quad (\mathrm{E_{iii}}) \textit{ if } n = 2j, \\ &= \bigcup_{i=0}^{j} S_{2i+1} && \textit{for} \quad (\mathrm{E_{iv}}) \textit{ if } n = 2j + 1. \end{aligned}$$

Suppose that Eq. (E) is nonoscillatory on $[a, \infty)$ and that $p \not\equiv 0$ on $[a_1, \infty)$ for every $a_1 \geqslant a$. It follows from [4, Corollary 3] that S_0 contains at most one linearly independent solution for $(\mathrm{E_{ii}})$ and $(\mathrm{E_{iii}})$. On the other hand, S_0 is known to contain at least one solution [2, 4, 10]. Therefore, S_0 contains exactly one linearly independent solution, S_0 is a one-dimensional linear space, for $(\mathrm{E_{ii}})$ and $(\mathrm{E_{iii}})$. In this connection we have the following results.

THEOREM 3. *Assume that* (E) *is nonoscillatory on* $[a, \infty)$ *and that condition* (1) *is satisfied. For* $(\mathrm{E_i})$ *and* $(\mathrm{E_{iv}})$, *the dimension of the linear space* S_1 *is at most* 2. *For* $(\mathrm{E_{ii}})$ *and* $(\mathrm{E_{iii}})$, *the dimension of the linear space* $S_0 \cup S_2$ *is at most* 3.

Proof. It is easily seen that S_1 and $S_0 \cup S_2$ are linear spaces in respective cases. Suppose that S_1 contains three linearly independent solutions y_1, y_2, and y_3 which are nonnegative on $[b, \infty)$. Without loss of generality, we may assume that $y_3 > y_2 > y_1$ on $[c, \infty)$, for some $c \geqslant b$. In this case,

$$\lim_{x\to\infty}(y_j(x)/y_k(x)) = \infty, \tag{13}$$

if $j > k$, $j, k = 1, 2, 3$. If this were not true, we would have the following two alternatives: There exists a constant $\gamma \geqslant 1$ such that either $y_j(x)/y_k(x)$ assumes the value γ an infinite number of times on $[c, \infty)$, or else

$$\lim_{x\to\infty}(y_j(x)/y_k(x)) = \gamma.$$

The first alternative is impossible because $y_j - \gamma y_k$ would be an oscillatory solution of (E). The second alternative is also impossible because it would imply that

$$\lim_{x\to\infty}(y_j(x) - \gamma y_k(x)) = 0,$$

contrary to the fact that S_0 is empty for (E_i) and (E_{iv}). Choose a constant $K > 0$ such that $u \equiv y_2 - Ky_1$ has a zero at some point $\xi \in [c, \infty)$. Let $\eta \geqslant \xi$ be the largest zero of u, i.e., $u(\eta) = 0$ and $u > 0$ on (η, ∞). Define $K_1 = \sup G$, where G is the set of real numbers $\beta \geqslant 1$ such that $y_3 - \beta u \geqslant 0$ on $[\eta, \infty)$. If $\tau \in (\eta, \infty)$, then $1 \leqslant \beta \leqslant y_3(\tau)/u(\tau)$, and consequently $1 \leqslant K_1 \leqslant y_3(\tau)/u(\tau)$. The solution $v \equiv y_3 - K_1 u \geqslant 0$ on $[\eta, \infty)$; more important, there exists a point $\zeta \in (\eta, \infty)$ at which $v(\zeta) = 0$. If there were no such point $\zeta \in (\eta, \infty)$, then $v \equiv y_3 - K_1 u > 0$ on $[\eta, \infty)$, i.e., $u/y_3 < 1/K_1$ on $[\eta, \infty)$. Since $u(x)/y_3(x) \to 0$ as $x \to \infty$ by (13), there exists an $\epsilon > 0$ such that $u/y_3 < 1/(K_1 + \epsilon)$ on $[\eta, \infty)$. This means that $y_3 - (K_1 + \epsilon)u > 0$ on $[\eta, \infty)$, contrary to the choice of K_1. Therefore, $v(\zeta) = 0$ for some $\zeta \in (\eta, \infty)$; moreover, $v'(\zeta) = 0$ since $v \geqslant 0$ in the neighborhood of ζ. But this result contradicts Lemma 1, according to which we must have $v' > 0$ on $[\eta, \infty)$, and proves that S_1 of (E_i) and (E_{iv}) is at most two-dimensional.

Suppose that $S_0 \cup S_2$ contains four linearly independent solutions y_i such that $y_i \geqslant 0$ on $[b, \infty)$, $i = 1, 2, 3, 4$, where $y_1 \in S_0$. Since S_0 for (E_{ii}) and (E_{iii}) is one-dimensional, we have $y_i \in S_2$, $i = 2, 3, 4$, and any linear combination of y_2, y_3, and y_4 again belongs to S_2. Suppose that $y_4 > y_3 > y_2 > y_1$ on $[c, \infty)$, for some $c \geqslant b$. Following the proof of (13), we can easily show that

$$\lim_{x\to\infty}(y_j(x)/y_k(x)) = \infty,$$

if $j > k$; $j, k = 1, 2, 3, 4$. Choose a positive constant K_2 such that $U \equiv y_2 - K_2 y_1$ has a zero at η_1 and $U > 0$ on (η_1, ∞). As we showed previously, there exists a constant $K_3 > 0$ such that $V \equiv y_3 - K_3 U$ is nonnegative on $[\eta_1, \infty)$ and $V(\zeta_1) = V'(\zeta_1) = 0$ for some $\zeta_1 \in (\eta_1, \infty)$. Similarly, by using the same argument once more, we may choose a constant $K_4 > 0$ such that $W \equiv y_4 - K_4 V$ is nonnegative on $[\zeta_1, \infty)$ and $W(\xi_1) = W'(\xi_1) = 0$ for some $\xi_1 \in (\zeta_1, \infty)$. Without loss of generality, we may assume that $y'_4(\zeta_1) > 0$ (for sufficiently large c). Since $W'(\zeta_1) = y'_4(\zeta_1) > 0$ and $W(\xi_1) = W'(\xi_1) = 0$, there exists $\tau \in (\zeta_1, \xi_1)$ such that $W''(\tau) = 0$. But this contradicts Lemma 1, which requires $W'' > 0$ on $[\zeta_1, \infty)$, and completes the proof.

Example. The differential equation

$$y^{(n)} + (\alpha/x^n)y = 0, \qquad \alpha \neq 0, \tag{14}$$

where α is a constant, satisfies condition (1), and has n linearly independent solutions of the form $y = x^\sigma$, where σ is a root of the algebraic equation

$$\sigma(\sigma - 1)(\sigma - 2) \cdots (\sigma - n + 1) + \alpha = 0. \tag{15}$$

It is easily seen that all the real roots of (15) lie in

$$\begin{array}{ll} (0, 1) \cup (2, 3) \cup \cdots \cup (n-2, n-1) & \text{if } n \text{ is even and } \alpha > 0, \\ (-\infty, 0) \cup (1, 2) \cup \cdots \cup (n-2, n-1) & \text{if } n \text{ is odd and } \alpha > 0, \\ (-\infty, 0) \cup (1, 2) \cup \cdots \cup (n-1, \infty) & \text{if } n \text{ is even and } \alpha < 0, \\ (0, 1) \cup (2, 3) \cup \cdots \cup (n-1, \infty) & \text{if } n \text{ is odd and } \alpha < 0. \end{array}$$

Therefore, this example illustrates Theorems 1 and 3 and Corollary 2. Moreover, it shows that the condition for disconjugacy stated in Corollary 1 is not a necessary condition, because Eq. (14) is known to be disconjugate on $[b, \infty)$, $b > 0$, if $|\alpha|$ is sufficiently small [3].

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